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Fuzzy Fixed Point Theorems in Ordered Cone Metric Spaces

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Abstract. In this article we introduce the notion of the multivalued fuzzy mappings satisfying w.l.b property and l.b properties and prove some results for multivalued generalize contractive fuzzy mappings in ordered-cone metric spaces without the assumption of normality on cones. We generalize many results in literature.

1. Introduction

The cone metric space is a generalization of metric space and is obtained by replacing the set of reals by an ordered Banach space. This idea was initiated in [23] by Huang and Zhang. Many generalizations of metric space have been presented on the behalf of this notion(see [1, 4–8, 17, 18, 20, 21, 24, 27, 28, 36, 37]). It is well known that the results concerning with non-normal cones in cone metric space are real generalizations of the results in metric space.

Partially ordered sets have many applications in computer languages, game theory, economics and many other fields [13]. The mathematical analysis under the domain of partially ordered sets evolved the fixed point theory to generalize and solve many results and problems in linear and non-linear analysis [2, 3, 15, 19, 25, 26, 32, 38]. In 2010 Altun and Damjanović generalized the results of [2] without the assumption of normality on cone and proved the following main result:

Theorem 1.1. *[3] Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $f : X \to X$ be a continuous and non-decreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following assertions hold:*

(i) there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + 2\beta + 2\gamma < 1$ such that $d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)]$ for all $x, y \in X$ with $y \sqsubseteq x$,

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$,

(iii) if an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.

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Then f has a fixed point $x \in X$.

Heilpern [22] introduced the notion of fuzzy mappings and presented a fixed point theorem for fuzzy mappings in complete metric linear spaces, that extended the Banach contraction principle and Nadler's [31] fixed point theorem. After that many other authors [10, 11, 14, 29, 30, 33–35, 39, 40] generalized this result and studied the existence of fixed points and common fixed points of fuzzy mappings satisfying a contractive type condition.

Recently Azam and Mehmood in [9] introduced the notion of the multivalued mappings having l.b and g.l.b properties in cone metric spaces and proved some new Kannan-type and Chatterjea-type results. In this article, we explore these investigations for the case of fuzzy mappings on a cone metric space endowed with a partial order on space. We introduce the set valued fuzzy mappings having weak lower bound property (w.l.b property) and lower bound property (l.b property) and prove the results for generalized contractions in ordered cone metric spaces. We obtain many corollaries and provide a nontrivial example to support our main theorem.

2. Preliminaries

Let $\mathbb{E}$ be a real Banach space with its zero element θ. A nonempty subset P of $\mathbb{E}$ is called a cone if
(i). P is nonempty closed and $P \neq \{\theta\}$.
(ii). $P \cap (-P) = \{\theta\}$;
(iii). if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$.

For a given cone $P \subseteq \mathbb{E}$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$; $x \prec y$ stands for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is said to be solid if it has a nonempty interior.

Now let us recall the following definitions and remarks:

Definition 2.1. *[23] Let X be a nonempty set. A vector-valued function $d : X \times X \to \mathbb{E}$ is said to be a cone metric if the following conditions hold:*
(C1) $\theta \preceq d(x, y)$ *for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;*
(C2) $d(x, y) = d(y, x)$ *for all $x, y \in X$;*
(C3) $d(x, z) \preceq d(x, y) + d(y, z)$ *for all $x, y, z \in X$. The pair (X, d) is then called a cone metric space.*

The cone metric d in X generate a topology τ_d. The base of topology τ_d consist of the sets

$$B_c(y) = \{x \in X : d(x, y) \ll c\} \text{ for some } c \in \mathbb{E} \text{ with } \theta \ll c.$$

For $x_0 \in X$ and $\theta \ll r$, we define closed ball

$$\bar{B}(x_0, r) := \{x \in X : d(x_0, x) \preceq r\},$$

in cone metric space (X, d). A set $A \subset (X, d)$ is called closed if, for any sequence $\{x_n\} \subset A$ converges to x, we have $x \in A$.

Definition 2.2. *[23] Let(X, d) be a cone metric space, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then*
(i) *$\{x_n\}$ converges to x if for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$;*
(ii) *$\{x_n\}$ is a Cauchy sequence if for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$;*
(iii) *(X, d) is complete if every Cauchy sequence in X is convergent.*

Remark 2.3. *[24] The results concerning fixed points and other results, in the case of cone spaces with non-normal solid cones, cannot be provided by reducing to metric spaces, because in this case neither of the conditions of lemmas 1-4 in [23] hold. Further, the vector cone metric is not continuous in the general case, i.e. from $x_n \to x$, $y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x, y)$.*

Let $\mathbb{E}$ be a Banach space with solid cone P. The following properties will be used very often (for more details, see [24, 41]).

($PT1$) If $u \leq v$ and $v \ll w$, then $u \ll w$.
($PT2$) If $u \ll v$ and $v \leq w$, then $u \ll w$.
($PT3$) If $u \ll v$ and $v \ll w$, then $u \ll w$.
($PT4$) If $\theta \leq u \ll c$ for each $c \in intP$, then $u = \theta$.
($PT5$) If $a \leq b + c$ for each $c \in intP$, then $a \leq b$.
($PT6$) If $a \leq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
($PT7$) If $c \in intP$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

Definition 2.4. *[2] A partially ordered set consists of a set X and a binary relation $\sqsubseteq$ on X which satisfies the following conditions:*
(i). $x \sqsubseteq x$ (reflexivity),
(ii). if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$ (antisymmetric),
(iii). if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$ (transitivity),
for all x, y and z in X. A set with a partial order $\sqsubseteq$ is called a partially ordered set. Let $(X, \sqsubseteq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be comparable elements of X if either $x \sqsubseteq y$ or $y \sqsubseteq x$.

Definition 2.5. *[12] Let A and B be two nonempty subsets of $(X, \sqsubseteq)$, the relations between A and B are denoted and defined as follows:*
(i). $A \sqsubseteq_1 B$: if for every $a \in A$ there exists $b \in B$ such that $a \sqsubseteq b$,
(ii). $A \sqsubseteq_2 B$: if for every $b \in B$ there exists $a \in A$ such that $a \sqsubseteq b$,
(iii). $A \sqsubseteq_3 B$: if for every $a \in A$, $b \in B$ implies $a \sqsubseteq b$.

A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α -level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1],$$

$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

Here $\overline{B}$ denotes the closure of the set B.

Definition 2.6. *Let X be an arbitrary set, Y be a cone metric space. A mapping F is called fuzzy mapping if F is a mapping from X into $\mathbb{F}(\mathbb{X})$. A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in $F(x)$.*

Definition 2.7. *Let (X, d) be a cone metric space and S, F be fuzzy mappings from X into $\mathbb{F}(\mathbb{X})$. A point $z \in X$ is called a fuzzy fixed point of F if $z \in [Fz]_\alpha$, where $\alpha \in [0, 1]$. The point $z \in X$ is called a common fuzzy fixed point of S and F if $z \in [Sz]_\alpha \cap [Fz]_\alpha$. When $\alpha = 1$, it is called a common fixed point of fuzzy mappings.*

3. Main results

Let (X, d) be a *cone* metric space with a cone P with non-empty interior and let $P_{cl}(X)$ be a collection of nonempty closed subsets of X. Let $F : X \to P_{cl}(X)$ be a multivalued map. For $x \in X, A \in P_{cl}(X)$, define

$$W_x(A) = \{d(x, a) : a \in A\}.$$

Thus, for $x, y \in X$,

$$W_x(Fy) = \{d(x, u) : u \in Fy\}.$$

Definition 3.1. *[17] Let (X,d) be a cone metric space with a cone P. A set-valued mapping $F : X \to 2^{\mathbb{E}}$ is called bounded from below if for all $x \in X$ there exists $z(x) \in \mathbb{E}$ such that*

$$Fx - z(x) \subset P.$$

Definition 3.2. *Let (X,d) be a cone metric space with a cone P. The fuzzy mapping $F : X \to \mathbb{F}(\mathbb{X})$ is said to have weak lower bound property (w.l.b property) on (X,d), if for each $x \in X$ there exists some $\alpha \in (0,1]$ such that, the multivalued mapping $F_x : X \to P_{cl}(X)$ defined by*

$$F_x(y) = W_x([Fy]_\alpha)$$

is bounded from below. That is, for $x, y \in X$ associated with $\alpha \in (0,1]$, there exists an element $wl_x([Fy]_\alpha) \in \mathbb{E}$ such that;

$$W_x([Fy]_\alpha) - wl_x\left([Fy]_\alpha\right) \subset P,$$

where $wl_x([Fy]_\alpha)$ is called lower bound of F called weak lower bound associated with (x,y). By $WL_{xy}(F)$ we denote the set of all weak lower bounds F associated with (x,y). Moreover, $\bigcup_{x,y\in X} WL_{xy}(F)$ is denoted by $WL_X(F)$.

Definition 3.3. *Let (X,d) be a cone metric space with a cone P. The fuzzy mapping $F : X \to \mathbb{F}(\mathbb{X})$ is said to have lower bound property (l.b property) on (X,d), if for each $x \in X$ and each $\alpha \in (0,1]$, the multivalued mapping $F_x : X \to P_{cl}(X)$ defined by,*

$$F_x(y) = W_x([Fy]_\alpha)$$

is bounded from below. That is, for $x, y \in X$ and $\alpha \in (0,1]$, there exists an element $l_x([Fy]_\alpha) \in \mathbb{E}$ such that;

$$W_x([Fy]_\alpha) - l_x\left([Fy]_\alpha\right) \subset P,$$

where $l_x([Fy]_\alpha)$ is called lower bound of F associated with (x,y). By $L_{xy}(F)$ we denote the set of all lower bounds F associated with (x,y). Moreover, $\bigcup_{x,y\in X} L_{xy}(F)$ is denoted by $L_X(F)$.

Remark 3.4. *Note that the fuzzy mapping $F : X \to \mathbb{F}(\mathbb{X})$ has l.b property if for $x, y \in X$, $l_x([Fy]_\alpha) \in \mathbb{E}$ exists for all $\alpha \in (0,1]$, . Whereas $F : X \to \mathbb{F}(\mathbb{X})$ has w.l.b property if for $x, y \in X$, $l_x([Fy]_\alpha) \in \mathbb{E}$ exists at least for one $\alpha \in (0,1]$, .*

According to [16], for $p \in \mathbb{E}$, let us denote

$$s(p) = \{q \in \mathbb{E} : p \preceq q\}$$

and

$$s(a,B) = \bigcup_{b\in B} s(d(a,b)) = \bigcup_{b\in B} \{x \in \mathbb{E} : d(a,b) \preceq x\} \text{ for } a \in X \text{ and } B \in P_{cl}(X).$$

For $A, B \in P_{cl}(X)$, we denote

$$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \cap \left(\bigcap_{b\in B} s(b,A)\right).$$

Let us recall the following lemma which will be used to prove our main Theorem.

Lemma 3.5. *[16, 41] Let (X,d) be a cone metric space with a cone P. Then we have:*
(i) Let $p, q \in \mathbb{E}$. If $p \preceq q$, then $s(q) \subset s(p)$.
(ii) Let $x \in X$ and $A \in \Lambda$. If $\theta \in s(x,A)$, then $x \in A$.
(iii) Let $q \in P$ and let $A, B \in \Lambda$ and $a \in A$. If $q \in s(A,B)$, then $q \in s(a,B)$.
(iv) For all $q \in P$ and $A, B \in \Lambda$. Then $q \in s(A,B)$ if and only if there exist $a \in A$ and $b \in B$ such that $d(a,b) \preceq q$.

Remark 3.6. *[16] Let (X, d) be a cone metric space. If $\mathbb{E} = R$ and $P = [0, +\infty)$, then (X, d) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A, B) = inf s(A, B)$ is the Hausdorff distance induced by d. Also, $s(\{x\}, \{y\}) = s(d(x, y))$ for all $x, y \in X$.*

Theorem 3.7. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ be a fuzzy mapping having w.l.b property.*

(i) If for $x, y \in X$ with $x \sqsubseteq y$ there exist $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + 2a_2 + 2a_3 < 1$, such that

$$a_1 d(x, y) + a_2[w\ell_x([Fx]_\alpha) + w\ell_y\left([Fy]_\alpha\right)] + a_3[w\ell_x\left([Fy]_\alpha\right) + w\ell_y([Fx]_\alpha)] \in s([Fx]_\alpha, [Fy]_\alpha).$$

(ii) There exists some $x_0 \in X, \alpha \in (0, 1)$ such that $\{x_0\} \sqsubseteq_1 [Fx_0]_\alpha$.
(iii) For $x, y \in X, x \sqsubseteq y$ implies $[Fx]_\alpha \sqsubseteq_3 [Fy]_\alpha$.
(iv) If an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.

Then there exists some $v \in X$ such that $v \in [Fv]_\alpha$.

Proof. Let x_0 be an arbitrary point in X. By using (ii) there exists some $x_1 \in [Fx_0]_\alpha$ such that $x_0 \sqsubseteq x_1$. Consider

$$a_1 d(x_0, x_1) + a_2[w\ell_{x_0}([Fx_0]_\alpha) + w\ell_{x_1}([Fx_1]_\alpha)] + a_3[w\ell_{x_0}([Fx_1]_\alpha) + w\ell_{x_1}([Fx_0]_\alpha)] \in s([Fx_0]_\alpha, [Fx_1]_\alpha).$$

By assumption (ii), $x_0 \sqsubseteq x_1$ implies $[Fx_0]_\alpha \sqsubseteq_3 [Fx_1]_\alpha$, using lemma 3.1 we can find some $x_2 \in [Fx_1]_\alpha$ satisying

$$a_1 d(x_0, x_1) + a_2[w\ell_{x_0}([Fx_0]_\alpha) + w\ell_{x_1}([Fx_1]_\alpha)] + a_3[w\ell_{x_0}([Fx_1]_\alpha) + w\ell_{x_1}([Fx_0]_\alpha)] \in s(d(x_1, x_2)) \text{ such that } x_1 \sqsubseteq x_2.$$

It implies that

$$d(x_1, x_2) \preceq a_1 d(x_0, x_1) + a_2[w\ell_{x_0}([Fx_0]_\alpha) + w\ell_{x_1}([Fx_1]_\alpha)] + a_3[w\ell_{x_0}([Fx_1]_\alpha) + w\ell_{x_1}([Fx_0]_\alpha)]$$

As

$$W_{x_i}(\left[Fx_j\right]_\alpha) - w\ell_{x_i}\left(\left[Fx_j\right]_\alpha\right) \subset P, \text{ for } i, j \in \{0, 1\}$$

It yields,

$$\begin{aligned}
w\ell_{x_0}([Fx_0]_\alpha) &\preceq d(x_0, x_1),\\
w\ell_{x_1}([Fx_0]_\alpha) &\preceq d(x_1, x_1),\\
w\ell_{x_0}([Fx_1]_\alpha) &\preceq d(x_0, x_2),\\
w\ell_{x_1}([Fx_1]_\alpha) &\preceq d(x_1, x_2),
\end{aligned}$$

thus

$$\begin{aligned}
d(x_1, x_2) &\preceq a_1 d(x_0, x_1) + a_2[d(x_0, x_1) + d(x_1, x_2)] + a_3[d(x_0, x_2) + d(x_1, x_1)]\\
&\preceq a_1 d(x_0, x_1) + a_2[d(x_0, x_1) + d(x_1, x_2)] + a_3[d(x_0, x_2) + \theta]\\
&\preceq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_2)\\
&\preceq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_1) + a_3 d(x_1, x_2)\\
&\preceq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} d(x_0, x_1).
\end{aligned}$$

By given condition $\frac{a_1+a_2+a_3}{1-a_2-a_3} < 1$, let $\frac{a_1+a_2+a_3}{1-a_2-a_3} = \eta$ then

$$d(x_1, x_2) \preceq \eta d(x_0, x_1).$$

By mathematical induction, we construct a sequence $\{x_n\}$ in X such that

$$d(x_n, x_{n+1}) \preceq \eta d(x_{n-1}, x_n),\ x_{n+1} \in Fx_n \text{ with } x_n \sqsubseteq x_{n+1} \text{ for } n = 1, 2, 3 \cdots.$$

Now for $m > n$, this gives

$$d(x_n, x_m) \leq \frac{\eta^n}{1-\eta} d(x_0, x_1).$$

Since $\eta^n \to 0$ *as* $n \to \infty$, this gives us $\frac{\eta^n}{1-\eta} d(x_0, x_1) \to \theta$ as $n \to \infty$. Now, according to $(PT7)$ and $(PT1)$, we can conclude that for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_1 such that $d(x_n, x_m) \ll c$ for all $m, n \geq n_1$, so $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, $\{x_n\}$ is convergent in X and $\lim_{n\to\infty} x_n = v$. Hence, for every $c \in \mathbb{E}$ with $\theta \ll c$, there is a natural number k_1 such that

$$\frac{1 + a_2 + a_3}{1 - a_2 - a_3} d(v, x_{n+1}) \ll c, \ \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} d(x_n, v) \ll c \ for\ all\ n \geq k_1. \tag{3}$$

We now show that $v \in [Fv]_\alpha$. By (iv) as $x_n \sqsubseteq v$ for all n, then consider

$$a_1 d(x_n, v) + a_2[w\ell_{x_n}([Fx_n]_\alpha) + w\ell_v([Fv]_\alpha)] + a_3[w\ell_{x_n}([Fv]_\alpha) + w\ell_v([Fx_n]_\alpha)] \in s([Fx_n]_\alpha, [Fv]_\alpha).$$

Using Lemma 3.5

$$a_1 d(x_n, v) + a_2[w\ell_{x_n}([Fx_n]_\alpha) + w\ell_v([Fv]_\alpha)] + a_3[w\ell_{x_n}(Fv) + w\ell_v([Fx_n]_\alpha)] \in s(x_{n+1}, [Fv]_\alpha)$$

So there exists $u_n \in [Fv]_\alpha$ satisfying

$$a_1 d(x_n, v) + a_2[w\ell_{x_n}([Fx_n]_\alpha) + w\ell_v([Fv]_\alpha)] + a_3[w\ell_{x_n}([Fv]_\alpha) + w\ell_v([Fx_n]_\alpha)] \in s(d(x_{n+1}, u_n)), \text{ such that } x_{n+1} \sqsubseteq u_n.$$

which implies that

$$\begin{aligned} d(x_{n+1}, u_n) &\leq a_1 d(x_n, v) + a_2[w\ell_{x_n}([Fx_n]_\alpha) + w\ell_v([Fv]_\alpha)] + a_3[w\ell_{x_n}([Fv]_\alpha) + w\ell_v([Fx_n]_\alpha)] \\ &\leq a_1 d(x_n, v) + a_2[d(x_n, x_{n+1}) + d(v, u_n)] + a_3[d(x_n, u_n) + d(v, x_{n+1})] \end{aligned}$$

Therefore, by (3),

$$\begin{aligned} d(v, u_n) &\leq d(v, x_{n+1}) + d(x_{n+1}, u_n) \\ &\leq d(v, x_{n+1}) + a_1 d(x_n, v) + a_2[d(x_n, x_{n+1}) + d(v, u_n)] + a_3[d(x_n, u_n) + d(v, x_{n+1})] \\ &\leq \frac{1 + a_2 + a_3}{1 - a_2 - a_3} d(v, x_{n+1}) + \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} d(x_n, v) \\ &\ll c, \text{ for all } n \geq k_1 = k_1(c). \end{aligned}$$

Hence $\lim_{n\to\infty} u_n = v$. Since $[Fv]_\alpha$ is closed so $v \in [Fv]_\alpha$. □

Example 3.8. *Let* $X = [0, 1]$, $\mathbb{E} = C_R^1[0, 1]$ *with norm* $\|f\| = \|f\|_\infty + \|f'\|_\infty$ *and*

$$P = \{x \in \mathbb{E} : x \geq 0, \ on\ [0, 1]\}.$$

Then, P is a non-normal solid cone. Define $d : X \times X \to \mathbb{E}$ *by*

$$(d(x, y))(t) = |x - y| e^t$$

where $0 \leq t \leq 1$*. Then d is a complete cone metric on X. Define a partial order on X, for* $x, y \in X$, $x \sqsubseteq y \Leftrightarrow d_1(x, y) \leq y - x$*, where* $d_1(x, y) = |x - y|$ *is complete metric on X. Consider a mapping*

$$F : X \to \mathbb{F}(\mathbb{X})$$

defined by

$$F(x)(t) = \begin{cases} \frac{\alpha}{3} & 0 \leqslant t < \frac{x}{16} \\ \frac{\alpha}{2} & \frac{x}{16} \leqslant t < \frac{x}{9} \\ \alpha & \frac{x}{9} \leqslant t \leqslant x \end{cases},$$

$$[Fx]_\alpha = [\frac{x}{9}, x].$$

Let for $x \sqsubseteq y$,

$$w\ell_x([Fx]_\alpha) \leq d(x,u) \text{ for } u \in [\frac{x}{9}, x], \text{ that is } w\ell_x([Fx]_\alpha) = \theta$$

$$w\ell_y\left([Fy]_\alpha\right) \leq d(y,u) \text{ for } u \in [\frac{y}{9}, y], \text{ that is } w\ell_y\left([Fy]_\alpha\right) = \theta$$

$$w\ell_x\left([Fy]_\alpha\right) \leq d(x,u) \text{ for } u \in [\frac{y}{9}, y],$$

$$w\ell_x\left([Fy]_\alpha\right) \leq d(x,u) = \begin{cases} \left|x - \frac{y}{9}\right| e^t \text{ if } x \leq \frac{5y}{9} \\ \left|x - y\right| e^t \text{ if } x > \frac{5y}{9} \end{cases}$$

$$\begin{aligned} w\ell_y([Fx]_\alpha) &\leq d(y,u) \text{ for } u \in [\frac{x}{9}, x], \\ w\ell_y([Fx]_\alpha) &\leq d(y,u) = \left|x - y\right| e^t. \end{aligned}$$

Also

$$s([Fx]_\alpha, [Fy]_\alpha) = s(\left|\frac{x}{9} - \frac{y}{9}\right| e^t).$$

Since

$$\left|\frac{x}{9} - \frac{y}{9}\right| e^t \leq a_1 d(x,y) = \frac{1}{2}\left|x - y\right| e^t,$$

so

$$\left|\frac{x}{9} - \frac{y}{9}\right| e^t \leq a_1 d(x,y) + a_2[w\ell_x([Fx]_\alpha) + w\ell_y\left([Fy]_\alpha\right)] + a_3[w\ell_x\left([Fy]_\alpha\right) + w\ell_y([Fx]_\alpha)]$$

Thus

$$a_1 d(x,y) + a_2[w\ell_x([Fx]_\alpha) + w\ell_y\left([Fy]_\alpha\right)] + a_3[w\ell_x\left([Fy]_\alpha\right) + w\ell_y([Fx]_\alpha)] \in s([Fx]_\alpha, [Fy]_\alpha).$$

Hence all the conditions of our main theorem are satisfied to obtain the fixed point of F.

For $a_2 = a_3 = 0$, and $a_1 < 1$, we have the following Nadler's type corollary.

Corollary 3.9. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ be a fuzzy mapping having w.l.b property.*
(i). If for $x, y \in X$ with $x \sqsubseteq y$ there exist $a_1 \in [0,1)$ and some $\alpha \in (0,1]$ such that

$$a_1 d(x,y) \in s([Fx]_\alpha, [Fy]_\alpha).$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 [Fx_0]_\alpha$.
(iii). For every $x \sqsubseteq y$ implies $[Fx]_\alpha \sqsubseteq_3 [Fy]_\alpha$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in [Fv]_\alpha$.

For $a_1 = a_3 = 0$, and $a_2 \in [0, \frac{1}{2})$, we have the following Kannan's type corollary.

Corollary 3.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ be a fuzzy mapping having w.l.b property.*
(i). If for $x, y \in X$ with $x \sqsubseteq y$ there exists $a_2 \in [0, \frac{1}{2})$, and some $\alpha \in (0, 1]$ such that

$$a_2[w\ell_x([Fx]_\alpha) + w\ell_y\left([Fy]_\alpha\right)] \in s([Fx]_\alpha, [Fy]_\alpha).$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 [Fx_0]_\alpha$.
(iii). For every $x \sqsubseteq y$ implies $[Fx]_\alpha \sqsubseteq_3 [Fy]_\alpha$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in [Fv]_\alpha$.

For $a_1 = a_2 = 0$, and $a_3 \in [0, \frac{1}{2})$, we have the following Chatterjea's type corollary.

Corollary 3.11. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ be a fuzzy mapping having w.l.b property.*
(i). If for $x, y \in X$ with $x \sqsubseteq y$ there exists $a_3 \in [0, \frac{1}{2})$, and some $\alpha \in (0, 1]$ such that

$$a_3[w\ell_x\left([Fy]_\alpha\right) + w\ell_y([Fx]_\alpha)] \in s([Fx]_\alpha, [Fy]_\alpha).$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 [Fx_0]_\alpha$.
(iii). For every $x \sqsubseteq y$ implies $[Fx]_\alpha \sqsubseteq_3 [Fy]_\alpha$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in [Fv]_\alpha$.

Corollary 3.12. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ be a fuzzy mapping having l.b property.*
(i). If for $x, y \in X$ with $x \sqsubseteq y$ there exist $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + 2a_2 + 2a_3 < 1$, such that

$$a_1 d(x, y) + a_2[\ell_x([Fx]_\alpha) + \ell_y\left([Fy]_\alpha\right)] + a_3[\ell_x\left([Fy]_\alpha\right) + \ell_y([Fx]_\alpha)] \in s([Fx]_\alpha, [Fy]_\alpha).$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 [Fx_0]_\alpha$.
(iii). For every $x \sqsubseteq y$ implies $[Fx]_\alpha \sqsubseteq_3 [Fy]_\alpha$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in [Fv]_\alpha$.

4. Applications

In the following we apply Fuzzy fixed point Theorem 3.7 to obtain fixed point of non-fuzzy(crisp) mutivalued mappings in partially ordered set with a complete metric on it.

Theorem 4.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $T : X \longrightarrow P_{cl}(X)$ be a multivalued mapping having l.b property.*
(i). If for $x, y \in X$ there exist $a, b, c \in [0, 1)$ with $a + 2b + 2c < 1$, such that

$$ad(x, y) + b[\ell_x(Tx) + \ell_y(Ty)] + c[\ell_x(Ty) + \ell_y(Tx)] \in s(Tx, Ty), \text{for all } x, y \in X \text{ with } x \sqsubseteq y.$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 Tx_0$.
(iii). For every $x \sqsubseteq y$ implies $Tx \sqsubseteq_3 Ty$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in Tv$.

Proof. Consider a fuzzy mappings $F : X \longrightarrow \mathbb{F}(\mathbb{X})$ defined by

$$Ux = \chi_{Tx},$$

where χ_{Tx} is Characteristic function of the closed set Tx. Then for $\alpha \in (0, 1]$, $say for(\alpha = 1)$

$$[Ux]_\alpha = Tx,$$

Since for $x, y \in X$,

$$\begin{aligned} s([Ux]_\alpha, [Uy]_\alpha) &= s(Tx, Ty), \\ \ell_x([Ux]_\alpha) &= \ell_x(Tx), \\ \ell_y\left([Uy]_\alpha\right) &= \ell_y(Ty), \\ \ell_x\left([Uy]_\alpha\right) &= \ell_x(Ty), \\ \ell_y([Ux]_\alpha) &= \ell_y(Tx). \end{aligned}$$

Therefore the Theorem 3.7 can be applied to obtain a point $v \in X$ such that $v \in T(v)$. □

Theorem 4.2. *Let (X, d) be a complete metric space. Let $T : X \longrightarrow CB(X)$ be a multivalued mapping.*
(i). If for there exist $a, b, c \in [0, 1)$ with $a + 2b + 2c < 1$, such that

$$H(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)], \text{for all } x, y \in X \text{ with } x \sqsubseteq y.$$

(ii). There exists some $x_0 \in X$ such that $\{x_0\} \sqsubseteq_1 Tx_0$.
(iii). For every $x \sqsubseteq y$ implise $Tx \sqsubseteq_3 Ty$.
(iv). If an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then there exists some $v \in X$ such that $v \in Tv$.

Remark 4.3. *By the suitable choice of a, b and c, one can obtain a number of corollaries of above theorem.*

If F is a single valued mapping then the following remarkable result is obtained which generalized many results in literature.

Corollary 4.4. *[3] Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete. Let $F : X \to X$ be a continuous and non-decreasing mapping w.r.t. $\sqsubseteq$. Suppose that the following assertions hold:*
(i) there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + 2\beta + 2\gamma < 1$ such that $d(Fx, Fy) \leq \alpha d(x, y) + \beta[d(x, Fx) + d(y, Fy)] + \gamma[d(x, Fy) + d(y, Fx)]$ for all $x, y \in X$ with $y \sqsubseteq x$,
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Fx_0$,
(iii) if an increasing sequence converges to x in X, then $x_n \sqsubseteq x$ for all n.
Then F has a fixed point $x \in X$.

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